On the LLN and CLT for Dyson Brownian motion

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Contents

Random matrices: Wishart

Random matrix theory dated back to Multivariate Statistical Analysis by Wishart in 1928.

Let $X_1, ..., X_n$ be independent $N(0, \Sigma)$-random vectors in $\mathbb{R}^p$, and let $X = [X_1, ..., X_n]$ be the $p \times n$ data matrix.

The distribution of a $p \times p$ random matrix

$$M = XX^T$$

is said the Wishart distribution with n degrees of freedom and covariance matrix Σ and is denoted by $W_p(n, \Sigma)$.

For $n \geq p$, the probability density function of M is

$$f(M) = \frac{1}{2^{np/2}\Gamma_p\left(\frac{n}{2}\right)|\Sigma|^{n/2}}|M|^{(n-p-1)/2}\exp\left(-\frac{1}{2}\mathrm{Tr}(\Sigma^{-1}M)\right)$$

with respect to Lebesque measure on the cone of symmetric positive definite matrices. Here, $\Gamma_p(\alpha)$ is the multivariate gamma function.

Random matrices: Hsu

P. Hsu 1939 gave an elegant proof of the joint distribution of eigenvalues of the Wishart matrix

$$M = XX^T.$$

P. Hsu: On the distribution of roots of certain determinantal equations, Ann. Eugenics, 9, 250-258, 1939.

See also Anderson (1957), An Introduction to Multivariate Statistical Analysis, 1984, 2003, Wiley.

Wigner

Though random matrices were first encountered in mathematical statistics by Hsu, Wishart, and others, intensively study of their properties in connection with nuclear physics began with the work of Wigner in the 1950s.

See Preface to the first edition, Random Matrices, Madan Lal Mehta

- In 1950, Wigner introduced the concept of statistical distribution of nuclear energy levels.
- In 1955, Wigner introduced ensembles of random matrices.
- In 1956, Wigner derived the famous semicircle law for the Wigner matrices.

Dyson

The mathematical foundations of random matrix theory were established in a series of beautiful papers by Dyson. He introduced the classification of random matrix ensembles according to their invariance properties under time reversal.

only three different possibilities exist: a system is not time reversal invariant, or a system is time reversal invariant with the square of the time reversal invariance operator either equal to 1 or -1.

The matrix elements of the corresponding random matrix ensembles are complex, real and self-dual quaternion, respectively,

The corresponding invariant Gaussian ensembles of Hermitian random matrices are known as the Gaussian unitary ensemble (GUE), the Gaussian orthogonal ensemble (GOE) and the Gaussian symplectic ensemble (GSE), in that order.

Hua

Random matrix theory, which was first formulated in mathematical statistics, continued to develop in mathematics independently of the developments in physics.

Important results with regard to the integration measure of invariant random matrix ensembles were obtained by Hua [1958]. His results of more than a decade of work are summarized in his book that appeared in 1958 but which remained largely unknown.

Figure: Hua Loo Keng

Figure: Hsu Pao-Lu

Matrix models

Unitary invariant random matrices ensemble with general potential V has been received a lot of attention in theoretic physics in connection with the so-called matrix models.

- Fernandez-Fröhlich-Sokal 1992, Brezin and Zee 1993.
- E. Witten, Two dimensional gravity and intersection theory on moduli space, Survey in Diff Geom. 1, 1991, 243-310.
- Kontsevich (CMP1992) used the (complex) partition function for the matrix model with $V(x) = x^3$ wrt the Gaussian measures on Hermitian random matrices

$$\mathbb{E}_{\mathrm{Gaussian}}\left[\exp(\sqrt{-1}x^3)\right]$$

  to prove Witten's conjecture in the intersection theory of the moduli space of curves.
- Guionnet, Large random matrices: Lectures on macroscopic asymptotics, Springer, 2008.

Unitary invariant ensemble

Let $V : \mathbb{R} \to [0, \infty)$ be a continuous function. Let $\mathcal{H}_N$ be the set of $N \times N$ Hermitian matrices. Consider the following probability distribution on $\mathcal{H}_N$

$$d\mu_N(M) = \frac{1}{Z_N} \exp(-N \mathrm{Tr} V(M)) dM.$$

Weyl integration formula (H. Weyl 1926): If $f : \mathcal{H}_N \to \mathbb{R}$ invariant under conjugation by unitary matrices, i.e. $f(UMU^*) = f(M)$, then

$$\int_{\mathcal{H}_N} f(M) d\mu_N(M) = \int_{\mathbb{R}^N} f(D(x)) \rho_N(x) dx,$$

where $D(x) = diag(x_1, \ldots, x_N)$ is the diagonal matrix of eigenvalues of M, and

$$\rho_N(x) = \frac{1}{Z_N} \prod_{i<j} |x_i - x_j|^2 \exp\left(-N \sum_{i=1}^{N} V(x_i)\right).$$

[H. Weyl (1926, 1946), L. Hsu(1939), L.K. Hua (1958), P. Deift (1999), P. J. Forrester (2005), Anderson-Guionnet-Zeitouni (2010), etc.]

Unitary invariant ensemble

Indeed, consider the matrix transformation

$$M = UDU^T,$$

where $D = \mathrm{diag}(x_1, \dots, x_N)$. By Weyl (1926, 1946), Hsu (1939), ... , Hua (1958), the Jacobian of the above matrix transformation is given Vandermant determinant

$$\Delta(x)^2 = \prod_{i<j} |x_i - x_j|^2$$

Thus, the joint distribution of the eigenvalues of M is given by

$$dP_N(x_1, \dots, x_N) = \frac{1}{Z_N} \prod_{i<j} |x_i - x_j|^2 \exp\left(-N \sum_{i=1}^{N} V(x_i)\right).$$

β-invariant ensemble

For $\beta \geq 1$, the β-invariant ensemble (log-gas model) has the following distribution density

$$\rho_N^\beta(x) = \frac{1}{Z_N^\beta} \prod_{1\leq i<j\leq N} |x_i - x_j|^\beta \exp(-\frac{\beta N}{2}\sum_{i=1}^N V(x_i)),$$

- $V(x) = \frac{x^2}{2}$, $\beta = 1$ (GOE), $\beta = 2$ (GUE), $\beta = 4$ (GSE).
- $V(x) = \frac{x^2}{2}$, $\beta = 8$, $N = 2$, octonions. See Forrester 2005 Book Log Gases and RM, S. Li Seminair de Probab. 2016.

The Voiculescu free entropy (also called free energy functional) is defined as follows

$$\begin{aligned} \Sigma_V(\mu) &= \lim_{N\to\infty} \frac{\log Z_N^\beta}{N^2} \\ &= -\frac{\beta}{2}\int_{\mathbb{R}^2} \log|x-y| d\mu(x)d\mu(y) + \int_{\mathbb{R}} V(x)d\mu(x). \end{aligned}$$

See Johansson, Biane, and also Ben Arous-Guionnet. In view of this, we have

$$\rho_N^\beta(x) = \frac{1}{Z_N^\beta}\exp(-\frac{\beta N^2}{2}\Sigma_V(\frac{1}{N}\sum_{i=1}^N \delta_{x_i})).$$

Equilibrium measure

Theorem (Boutet de Monvel-Pastur-Shcherbina 95, Johansson 98)

Suppose $V : \mathbb{R} \to [0, \infty)$ *is continuous and* $\exists\ \delta > 0$ *such that*

$$V(x) \geq (1+\delta)\log(x^2+1), \quad x >> 1. \tag{1}$$

Then $\exists!\ \mu_V \in \mathscr{P}(\mathbb{R})$ *with compact support such that*

$$\inf_{\mu \in \mathscr{P}(\mathbb{R})} \Sigma_V(\mu) = \Sigma_V(\mu_V),$$

Moreover, as $N \to \infty$, *it holds*

$$\mathbb{E}_N^\beta \left[\frac{1}{N} \sum_{i=1}^N \delta_{x_i} \right] \to \mu_V.$$

Indeed, the Euler-Lagrange equation of Σ_V implies that μ_V is the unique solution to

$$\mathrm{H}\mu_V(x) = \mathrm{P.V.} \int_{\mathbb{R}} \frac{d\mu_V(y)}{x-y} = \frac{1}{2} V'(x), \quad \forall\ x \in \mathrm{supp}\ \mu_V,$$

where

$$\mathrm{P.V.} \int_{\mathbb{R}} \frac{\mu_V(y)}{x-y} dy$$

is the Hilbert transform of μ_V.

The minimizer μ_V of Σ_V is called the equilibrium measure for the β-ensemble with potential V.

Wigner's theorem

In particular, for $\beta = 2$ and $V(x) = \frac{1}{2}x^2$, that is, GUE, then μ_V is the famous Wigner semicircle law.

Theorem (Wigner 1956)

For GUE, we have

$$\mathbb{E}_N\left[\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}\right] \to \mu_{sc}.$$

where μ_{sc} is the semicircle law

$$\mu_{sc}(dx) = \frac{1}{2\pi}\sqrt{4-x^2}1_{|x|\leq 2}dx.$$

For further work, see Bai-Silverstein (2010). Spectral Analysis of Large Dimensional Random Matrices. Springer.

Figure: E. Wigner

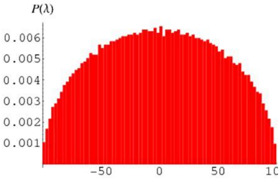

Figure: E. Wigner's semicircle law

Dyson Brownian motion for Log-gas

F. Dyson 1962: Dyson Brownian motion is the process of the eigenvalues of the Hermitian Brownian motion.

Theorem (Dyson 1962)

For any value of $\beta > 0$, if $\lambda_N(0) = (\lambda_N^1(0), \ldots, \lambda_N^N(0)) \in \triangle_N$. Then there exists a unique strong solution $(\lambda_N(t))_{t\geq 0}$ to the following SDE

$$d\lambda_N^i(t) = \sqrt{\frac{2}{\beta N}} dW_t^i + \frac{1}{N} \sum_{j:j\neq i} \frac{1}{\lambda_N^i(t) - \lambda_N^j(t)} dt,$$

with initial condition $\lambda_N(0)$ such that $\lambda_N(t) \in \triangle_N$ for all $t \geq 0$.

- The solution $(\lambda_N(t))_{t\geq 0}$ is called Dyson's Brownian motion.

Figure: F. Dyson

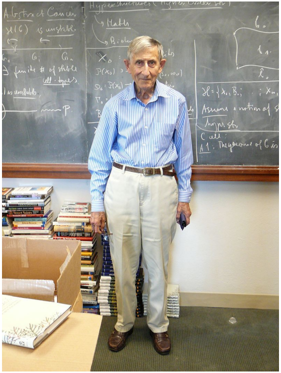

Figure: F. Dyson2

Freerman Dyson: Birds and frogs

Some mathematicians are birds, others are frogs. Birds fly high in the air and survey broad vistas of mathematics out to the far horizon. They delight in concepts that unify our thinking and bring together diverse problems from different parts of the landscape. Frogs live in the mud below and see only the flowers that grow nearby. They delight in the details of particular objects, and they solve problems one at a time. I happen to be a frog, but many of my best friends are birds. The main theme of my talk tonight is this. Mathematics needs both birds and frogs. Mathematics is rich and beautiful because birds give it broad visions and frogs give it intricate details. Mathematics is both great art and important science, because it combines generality of concepts with depth of structures. It is stupid to claim that birds are better than frogs because they see farther, or that frogs are better than birds because they see deeper. The world of mathematics is both broad and deep, and we need birds and frogs working together to explore it.

Notice of AMS 2009 (Einstein Lecture October 2008 cancelled)

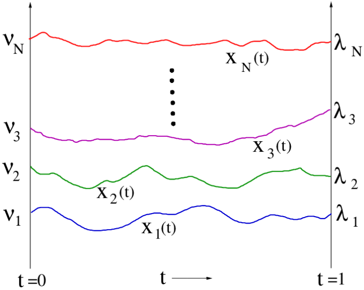

Figure: Dyson Brownian motion

Interacting particle system arising in non-equilibrium statistical mechanics

Consider SDE for N-particles $X_N^i(t) \in \mathbb{R}^n$ with interaction $W \in C^1(\mathbb{R}^n \setminus \{0\})$ and external potential $V \in C^1(\mathbb{R}^n)$:

$$\begin{aligned} dX_N^i(t) &= \sum_j \sigma_{ij}(X_N^i(t))dB_t^i - \nabla V(X_N^i(t))dt \\ &\quad - \sum_{j \neq i} \nabla W(X_N^i(t) - X_N^j(t))dt, \end{aligned}$$

where $B = (B^1, \ldots, B_N)$ is a Brownian motion on $\mathbb{R}^N$.
In particular, the following two interaction functions are the most difficult cases:

- Coulumb interaction $\quad W(x) = \frac{C_n}{|x|^{n-2}}$, $n \geq 3$, where

$$C_n = \frac{\Gamma(n/2)}{2\pi^{n/2}(n-2)}.$$

- Logarithmic Coulumb interaction $\quad W(x) = \log |x|^{-1}$, $n = 1, 2$.

Generalized Dyson Brownian motion

The purpose of this work is to study the generalized Dyson BM, the law of large numbers, the functional central limit theorem of its empirical measures and the longtime asymptotic beahvior.

First, we introduce the $(\mathrm{GDBM})_{\mathrm{V}}$ with generic potentials through matrix-valued diffusion process.

For $\beta = 1, 2, 4$, the matrix-valued diffusion process $X_t^{N,\beta}$ by solving SDE

$$dX_t^{N,\beta} = \sqrt{\frac{2}{\beta N}} dB_t^N - N\nabla \mathrm{Tr}\, V(X_t^{N,\beta})dt, \tag{2}$$

where B_t^N denotes the $N \times N$ Hermitian Brownian motion.

The eigenvalues process $\lambda_N^1(t), \ldots, \lambda_N^N(t)$ of $X_t^{N,\beta}$ is called $(\mathrm{GDBM})_{\mathrm{V}}$.

From matrix-valued diffusion process to eigenvalues process

Theorem (Li-Li-Xie JSP2020)

Let $V : \mathbb{R} \to \mathbb{R}$ be a real analytic function. Let $X_t^{N,\beta}$ be the matrix-valued diffusion process defined by (2) with $\beta = 1, 2$. Then, the eigenvalues $\lambda_N(t) = (\lambda_N^1(t), \dots, \lambda_N^N(t))$ of $X_t^{N,\beta}$ satisfy the SDE

$$d\lambda_N^i(t) = \sqrt{\frac{2}{\beta N}} dW_t^i + \frac{1}{N} \sum_{j:j\neq i} \frac{1}{\lambda_N^i(t) - \lambda_N^j(t)} dt - \frac{1}{2} V'(\lambda_N^i(t)) dt, \tag{3}$$

where the $W = (W^1, \dots, W_N)$ is a Brownian motion on $\mathbb{R}^N$.

- $V'(x) \equiv 0$, Dyson Brownian motion [Dyson, A Brownian-motion model for the eigenvalues of a random matrix. J. Math. Phys.1962].
- $V(x) = \frac{x^2}{2}$, Dyson Ornstein-Uhlenbeck Brownian motion [T. Chan 1992, Rogers-Shi 1993,Cépa and Lépingle 1997, Fontbona 2004, Guionnet 2008, Anderson, Guionnet and Zeitouni 2010].
- Using Dyson BM, T. Chan (PTRF 1992), Rogers-Shi (PTRF 1993) gave a dynamical proof of the Wigner theorem. See also Anderson-Guionnet-Zeitouni (2010) and Guionnet (2008).
- For $\beta = 4$, the eigenvalues $\lambda_{2N}(t) = (\lambda_{2N}^1(t), \dots, \lambda_{2N}^{2N}(t))$ of $X_t^{N,4}$ satisfy the same SDE of (6).

Dyson Brownian motion for general external potential

Theorem (Li-Li-Xie JSP2020)

Let V be a C^1 function satisfying the following conditions
(i) For all $R > 0$, $\exists K_R > 0$, such that $\forall x, y \in \mathbb{R}$ with $|x|,\ |y| \leq R$,

$$(x-y)(V'(x)-V'(y)) \geq -K_R|x-y|^2. \tag{4}$$

(ii) There exists a constant $C > 0$ such that

$$-xV'(x) \leq C(1+|x|^2), \quad \forall\, x \in \mathbb{R}. \tag{5}$$

Then, for all $\beta \geq 1$, and for any given $X_N(0) \in \triangle_N$, there exists a unique strong solution $(X_N(t))_{t\geq 0}$ taking values in $\triangle_N$ such that

$$dX_N^i(t) = \sqrt{\frac{2}{\beta N}}dW_t^i + \frac{1}{N}\sum_{j\neq i}\frac{1}{X_N^i(t)-X_N^j(t)}dt - \frac{1}{2}V'(X_N^i(t))dt. \tag{6}$$

where W is a Brownian motion on $\mathbb{R}^N$, $\Delta_N = \{(x_i)_{1\leq i\leq N} \in \mathbb{R}^N : x_1 < x_2 < \ldots < x_{N-1} < x_N\}$.

Dyson Brownian motion for general external potential

By Ito's calculus, for all $f \in C_b^2(\mathbb{R})$, we have

$$\begin{aligned} d\langle L_N(t), f\rangle &= \frac{1}{N}\sqrt{\frac{2}{\beta N}}\sum_{i=1}^N f'(\lambda_N^i(t))dW_t^i - \left\langle L_N(t), V'f'\right\rangle dt \\ &\quad + \frac{1}{2N}\left(\frac{2}{\beta}-1\right)\left\langle L_N(t), f''\right\rangle dt + \frac{1}{2}\int_{\mathbb{R}^2}\frac{f'(x)-f'(y)}{x-y}(L_N(t,dx)L_N(t,dy)dt. \end{aligned}$$

Let

$$M_N^f(t) := \frac{1}{N}\sqrt{\frac{2}{\beta N}}\sum_{i=1}^N f'(\lambda_N^i(t))dW_t^i.$$

Then M is a continuous $\mathcal{F}_t$-martingale with quadratic variation

$$\langle M_N^f\rangle_t = \frac{2}{\beta N^3}\sum_{i=1}^N \int_0^t |f'(\lambda_N^i(t))|^2 dt \le \frac{2T}{\beta N^2}\max_x |f'(x)|^2.$$

This derives that, if μ_t is a weak convergence limit of a subsequence of $L_N(t) \to \mu_t$, then we have

$$\frac{d}{dt}\langle \mu_t, f\rangle = \frac{1}{2}\int_{\mathbb{R}^2}\frac{f'(x)-f'(y)}{x-y}d\mu_t(x)d\mu_t(y) - \left\langle \mu_t, V'f'\right\rangle.$$

Dyson Brownian motion for general external potential

More generally, we can consider generalized Dyson Brownian motion (GDBM)

$$dx_t^i = \sigma_N(x_t^i)dW_t^i + \frac{1}{N}\sum_{j\neq i}\frac{1}{x_t^i - x_t^j}dt - V'(x_t^i)dt,$$

and consider the empirical measure

$$L_N(t) = \frac{1}{N}\sum_{i=1}^{N}\delta_{x_t^i}.$$

Assume that $\sigma_N \to \sigma$. Then $\{L_N(t), t\in[0,T]\}$ is tight in $C([0,T],\mathscr{P}(\mathbb{R}))$, and its subsequence limits satisfy the nonlinear McKean-Vlasov equation of the form

$$\frac{d}{dt}\mu_t(f) = \frac{\sigma^2}{2}\Delta\mu_t(f) + \frac{1}{2}\int_{\mathbb{R}^2}\frac{\partial_x f(x) - \partial_y f(y)}{x-y}\mu_t(dx)\mu_t(dy) - \int_{\mathbb{R}} V'(x)f'(x)\mu_t(dx).$$

See

- D. A. Dawson, J. Gärtner, Large deviations from the McKean-Vlasov limit for weakly interacting diffusions, Stochastics 20 (1987), 247-308.
- J. Gärtner, On the McKean-Vlasov limit for interacting diffusions, Math. Nachr. 137 (1988), 197-248.

Existence of McKean-Vlasov limit as $N \to \infty$

Denote by

$$L_N(t) = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_N^i(t)} \in \mathscr{P}(\mathbb{R}),$$

the spectral measure(also called empirical measure) of $\lambda_N(t)$.

Theorem (L-L-X JSP2020)

Let V be a C^2 function satisfying (1), (4) *and* (5). *Suppose*

$$\sup_{N \geq 0} \int_{\mathbb{R}} \log(x^2 + 1) dL_N(0) < \infty,$$

and

$$L_N(0) \to \mu \in \mathscr{P}(\mathbb{R}) \quad \text{as} \quad N \to \infty.$$

Then, the family $\{L_N(t), t \in [0, T]\}$ is precompact in $\mathcal{C}([0, T], (\mathscr{P}(\mathbb{R}), \text{weak topo}))$. Moreover, the limit of any weakly convergent subsequence of $\{L_N(t), t \in [0, T]\}$ is a weak solution of the McKean-Vlasov equation, i.e., for all $f \in C_b^2(\mathbb{R})$, $t \in [0, T]$,

$$\frac{d}{dt} \int_{\mathbb{R}} f(x) \mu_t(dx) = \frac{1}{2} \int_{\mathbb{R}^2} \frac{\partial_x f(x) - \partial_y f(y)}{x - y} \mu_t(dx) \mu_t(dy) - \frac{1}{2} \int_{\mathbb{R}} V'(x) f'(x) \mu_t(dx).$$

Convergence of McKean-Vlasov equation

The above result shows that $\{L_N(t), t \in [0,T]\}$ is tight in $C([0,T], \mathscr{P}(\mathbb{R}))$. It is natural to ask

Question

$$L_N(t) \to \mu(t) \quad ?$$

Let

$$G_t(z) = \int_{\mathbb{R}} \frac{\mu_t(dx)}{z-x}$$

be the Stieltjes transform of μ_t. Then $G_t(z)$ satisfies the following equation

$$\frac{\partial}{\partial t} G_t(z) = -G_t(z) \frac{\partial}{\partial z} G_t(z) - \frac{1}{2} \int_{\mathbb{R}} \frac{V'(x)}{(z-x)^2} \mu_t(dx). \tag{7}$$

Convergence of McKean-Vlasov equation

In particular, in the case $V(x) = x^2$, since

$$-\int_{\mathbb{R}} \frac{x}{(z-x)^2}\mu_t(dx) = z\frac{\partial}{\partial z}G_t(z) + G_t(z),$$

the Stieltjes transform of μ_t satisfies the complex Burgers equation

$$\frac{\partial}{\partial t}G_t(z) = \left(-G_t(z) + z\right)\frac{\partial}{\partial z}G_t(z) + G_t(z). \tag{8}$$

Chan (1992) and Rogers-Shi (1993) proved that the complex Burgers equation (8) has a unique solution, and the $t \to \infty$ limit exists and

$$\lim_{t\to\infty} G_t(z) = G_{\mu_{sc}}(z)$$

where $G_{sc}(z)$ is the Stieltjes transform of the Wigner semi-circle law μ_{sc}.

This gave a dynamic proof of the Wigner's theorem, i.e., $L_N(\infty)$ weakly converges to μ_{sc}.

See also Guionnet and Anderson-Guionnet-Zeitouni's books.

Convergence of McKean-Vlasov equation

However, for non quadratic potential V, $\int_{\mathbb{R}} \frac{V'(x)}{(z-x)^2}\mu_t(dx)$ in (7) cannot be expressed in terms of $G_t(z)$ and its derivatives with respect to z.

Thus, one cannot derive an analogue of the complex Burgers equation (8) for non quadratic potential V, and we need to find a new approach to prove the uniqueness of the weak solutions of the Mckean-Vlasov equation for general potential V.

We need a new method. In next pages, we use the theory of gradient flow on the Wasserstein space $\mathscr{P}_2(\mathbb{R})$ and the optimal transportation theory to study this problem.

McKean-Vlasov equation

The McKean-Vlasov equation for μ_t reads: for all $f \in C_b^2(\mathbb{R})$,

$$\frac{\partial}{\partial t}\mu_t(f) = \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{f'(x) - f'(y)}{x - y} d\mu_t(x) d\mu_t(y) - \frac{1}{2}\mu_t(f'V'). \tag{9}$$

In the case $\mu_t << dx$, and denoting $\rho_t = \frac{d\mu_t}{dx}$, then Integrating by parts show that, ρ satisfies the evolution equation

$$\frac{\partial \rho_t}{\partial t} = \frac{\partial}{\partial x}\left(\rho_t\left(\frac{1}{2}V' - \mathrm{H}\rho_t\right)\right), \tag{10}$$

where

$$\mathrm{H}\rho(x) = \mathrm{P.V.}\int_{\mathbb{R}} \frac{\rho(y)}{x - y} dy$$

is the Hilbert transform of ρ.

Otto's infinite dimensional Riemannian geometry on Wasserstein space

To study the uniqueness and the longtime behavior of the nonlinear Fokker-Planck equation (10), we first recall Otto's infinite dimensional Riemannian structure on the Wasserstein space $\mathscr{P}_2(\mathbb{R}^d)$.

Fix $fdx \in \mathscr{P}_2(\mathbb{R}^d)$, the tangent space of $\mathscr{P}_2(\mathbb{R}^d)$ at fdx is given by

$$T_{fdx}\mathscr{P}_2(\mathbb{R}^d) = \{s(x)dx : s \in W^{1,2}(\mathbb{R}^d, \mathbb{R}), \quad \int_{\mathbb{R}} sdx = 0\}.$$

By Otto (2001), for all $s_i dx \in T_{fdx}\mathscr{P}_2(\mathbb{R}^d)$, $i = 1, 2$, there exist a unique $\nabla p_i \in W^{1,2}(\mathbb{R}^d, \mathbb{R}^d)$, $i = 1, 2$, such that

$$s_i = -\nabla.(f\nabla p_i)$$

In view of this, Otto's infinite dimensional Riemannian metric on $T_{fdx}\mathscr{P}_2(\mathbb{R}^d)$ is defined by

$$g_{fdv}(s_1, s_2) = \int_{\mathbb{R}^d} \nabla p_1 \cdot \nabla p_2 fdx.$$

Voiculescu entropy and entropy dissipation formula

Inspired by Biane-Speicher AIHP 2001, and Carrillo-McCann-Villani RMI 2003 we obtained the following

Theorem (LLX JSP2020)

Under the condition (5) *for* C^2 *function* V, *the McKean-Vlasov equation* (10) *for* μ_t *is indeed the gradient flow of the Voiculescu free entropy* $\Sigma_V(\mu)$ *on the Wasserstein space on* $\mathscr{P}_2(\mathbb{R})$, *i.e.,*

$$\partial_t \rho = -\nabla \cdot \left(\rho \nabla \frac{\delta \Sigma_V}{\delta \rho} \right). \tag{11}$$

Indeed,

$$\frac{\delta \Sigma_V}{\delta \rho}(x) = V(x) - 2 \int_{\mathbb{R}} \log |x-y| \rho(y) dy.$$

Free relative entropy (Voiculescu, Biane)

$$\Sigma_V(\mu_t|\mu_V) = \Sigma_V(\mu_t) - \Sigma_V(\mu_V).$$

Free Fisher information (Voiculescu, Biane)

$$\mathrm{I}(\rho) = \int_{\mathbb{R}} \left[V'(x) - 2(\mathrm{H}\rho)(x) \right]^2 \rho(x) dx.$$

Voiculescu entropy and entropy dissipation formula

Theorem (LLX JSP2020)

Let $\xi = V' - 2\mathrm{H}\rho$. *We have*

$$\begin{aligned} \frac{d}{dt}\Sigma_V(\mu_t|\mu_V) &= -2\int_{\mathbb{R}}\left[V'(x) - 2(\mathrm{H}\rho)(x)\right]^2\rho(x)dx, \\ \frac{d^2}{dt^2}\Sigma_V(\mu_t|\mu_V) &= 2\int_{\mathbb{R}} V''(x)|V'(x) - 2\mathrm{H}\rho(x)|^2\rho(x)dx \\ &\quad + \int_{\mathbb{R}^2}\frac{\left[V'(x) - V'(y) - 2(\mathrm{H}\rho(x) - \mathrm{H}\rho(y))\right]^2}{(x-y)^2}\rho(x)\rho(y)dxdy. \end{aligned}$$

Theorem (LLX JSP2020)

Suppose that $V \in C^2(\mathbb{R}, \mathbb{R}^+)$ *and there exists a constant* $K \in \mathbb{R}$ *such that* $V'' \geq K$. *Then*

$$\frac{d^2}{dt^2}\Sigma_V(\mu_t|\mu_V) \geq K.$$

A heuristic proof was given in arxiv2013/2014 using Carrillo-McCann-Villani's approach, the rigorous proof was given in JSP2020 based on Biane-Speicher's a priori estimates to the above McKean-Vlasov equation: $\exists M, K_1, K_2 > 0$ which are independent of t such that

$\mathrm{supp}\mu_t \in [-M, M]$, $\|\rho(t)\|_\infty \leq \frac{K_1}{t} + K_2$, and $\|D^{1/2}\rho(t)\|_2 \leq \frac{K_1}{t} + K_2$.

Uniqueness of the McKean-Vlasov equation

The uniqueness of weak solutions to the M-V equation with logarithmic Coulomb interaction is a difficult problem. There was no result in the literature except fo quadratic potentials.

Using infinite dimensional geometry on the Wasserstein space $\mathscr{P}_2(\mathbb{R})$, and the theory of gradient flows in OPT developed by Otto, Otto-Villani, ..., and Ambrosio-Gigli-Savare, we were able to prove the uniqueness of weak solutions to the McKean-Vlasov equation (9) for general potentials with $V" \geq K$.

Theorem (LLX JSP2020)

Suppose that V be a C^2 function satisfying (4) *and* (5)*, and there exists a constant $K \in \mathbb{R}$ such that*

$$V''(x) \geq K, \quad \forall\, x \in \mathbb{R}.$$

Let $\mu_i(t)$ be two solutions of the McKean-Vlasov equation (9) *with initial data $\mu_i(0)$, $i = 1, 2$.*

Then for all $t > 0$, we have

$$W_2(\mu_1(t), \mu_2(t)) \leq e^{-Kt} W_2(\mu_1(0), \mu_2(0)).$$

In particular, the McKean-Vlasov equation (9) *has a unique weak solution.*

Sketch of proof: Key points

The proof is inspired by Otto (2003), Otto-Villani (2000), Carrillo-MacCann-Villani (2003), Villani (2009), and Ambrosio-Gigli-Savare (2005), and uses the following

Theorem (Blower2004, LLX JSP2020)

Assuming that $V \in C^2(\mathbb{R}, \mathbb{R}^+)$ and there exists a constant $K \in \mathbb{R}$ such that $V'' \geq K$. Then

$$\mathrm{Hess}_{\mathscr{P}_2(\mathbb{R})} \Sigma_V(\mu) \geq K.$$

We proved this result in August 2012 and noticed later (2013) from Villani's 2nd book Optimal Transport Old and New that Blower (2004) already proved the K-convexity of the Voiculescu entropy.

Weak Law of Large Numbers

By the precompactness of $L_N(t)$ and the uniqueness of the weak solution of the McKean-Vlasov equation (9), we have the Weak Law of Large Numbers for the empirical measures of the generalized Dyson Brownian motion.

Theorem (LLX JSP2020)

Suppose that $L_N(0)$ weakly converges to $\mu(0) \in \mathscr{P}(\mathbb{R})$. Let V be a C^2 function satisfying (1), (4) and (5) and $V'' \geq K$ for some constant $K \in \mathbb{R}$.

Then the empirical measure $L_N(t)$ weakly converges to the unique solution μ_t of the McKean-Vlasov equation (9) (equivalently (11)).

Moreover, for all $p \in [1, 2)$, we have

$$W_p(\mathbb{E}[L_N(t)], \mu_t) \to 0 \quad \text{as} \quad N \to \infty,$$

where the convergence is uniformly with respect to $t \in [0, T]$ for all fixed $T > 0$.

Recently, I extend this to all $p > 1$.

A comment

Results for the McKean–Vlasov equation were first established by Chan [12] and Rogers and Shi [49], who showed the existence of a solution for quadratic potentials V . The McKean–Vlasov equation for general potentials V was studied in detail in the works of Li, Li, and Xie. In their works [44] and [45], it was shown that, under very weak conditions on V , the solution of the McKean–Vlasov equation would converge to an equilibrium distribution for times $t >> 1$. The authors were able to interpret the time evolution under the McKean–Vlasov equation as a type of gradient descent on the space of measures. This gives the complete description of the Dyson Brownian motion on the macroscopic scale.

See Arka Adhikari, Jiaoyang Huang, Dyson Brownian motion for general and potential at the edge, Probability Theory and Related Fields (2020) 178:893–950.

Propagation of chaos: M. Kac 1954

Sznitman and Tanaka (1984): For exchangeable systems, Propagation of chaos is equivalent to the LLN for the empirical measure of the system.

Theorem (Li-Li-Xie JSP2020)

Assume the conditions in the above Theorem holds. Let $M_{N;k}(t; dx_1, \cdots, dx_k)$ *be the k-th moment measure for the random probability measure* $L_N(t, \cdot)$*, that is, for any Borel sets* $A_1, \cdots, A_k$,

$$M_{N;k}(t; A_1, \cdots, A_k) := \mathbb{E}(L_N(t, A_1) \cdots L_N(t, A_k)).$$

Then we have

$$\lim_{N\to\infty} \int_{\mathbb{R}^k} \varphi(x_1, \cdots, x_k) M_{N;k}(t; dx_1, \cdots, dx_k) = \int_{\mathbb{R}^k} \varphi(x_1, \cdots, x_k) \mu_t(dx_1) \cdots \mu_t(dx_k)$$

for any continuous, bounded φ *on* $\mathbb{R}^k$.

Extension of HWI

We also extend Otto-Villani's HWI inequality to the Voiculescu free entropy Σ_V, the W_2-Wasserstein distance and the free Fisher information I_V.

Theorem (Li-Li-Xie JSP2020)

Suppose that there exists a constant $K \in \mathbb{R}$ such that

$$V''(x) \geq 2K, \quad \forall x \in \mathbb{R}.$$

Let $\mu_i \in \mathscr{P}_2(\mathbb{R})$, $i = 1, 2$. Then for all $t > 0$, the HWI inequality holds

$$\Sigma_V(\mu_1) - \Sigma_V(\mu_2) \leq W_2(\mu_1, \mu_2)\sqrt{\mathrm{I}_V(\mu_1)} - \frac{K}{2} W_2^2(\mu_1, \mu_2). \tag{12}$$

In particular, for any weak solution to the McKean-Vlasov equation (11)*, we have*

$$\Sigma_V(\mu_t) - \Sigma_V(\mu_V) \leq W_2(\mu_t, \mu_V)\sqrt{\mathrm{I}_V(\mu_t)} - \frac{K}{2} W_2^2(\mu_t, \mu_V), \tag{13}$$

where

$$\mathrm{I}_V(\mu) := \int_{\mathbb{R}} |V'(x)/2 - \mathrm{H}\mu(x)|^2 d\mu(x).$$

Question

In the case $V(x) = \frac{x^2}{2}$, Rogers and Z. Shi (PTRF1993), Chan (PTRF 1992) proved that

$$\mu_t \to \mu_{sc} \quad \text{as} \quad t \to \infty.$$

It is natural to ask
Question: Under which condition on the potential V, it holds

$$\mu_t \to \mu_V \quad \text{as} \quad t \to \infty$$

in the weak convergence topology or with respect to the W_2-Wasserstein distance?
That is can we prove the following commutative diagram:

$$\begin{array}{ccc} L_N(t) & \Longrightarrow & \mu_t \\ \Downarrow & & \Downarrow ? \\ L_N & \Longrightarrow & \mu_V. \end{array} \tag{14}$$

If this is true, then with respect to the weak convergence on $\mathscr{P}(\mathbb{R})$ or the W_2-Wasserstein topology on $\mathscr{P}_2(\mathbb{R})$, we have

$$\lim_{N\to\infty} \lim_{t\to\infty} L_N(t) = \lim_{t\to\infty} \lim_{N\to\infty} L_N(t).$$

Convex potential

Using the HWI inequality and argument in optimal transportation theory, we proved the following

Theorem (Li-Li-Xie JSP2020)

Suppose that V is C^2-convex, i.e., $V'' \geq 0$. Then, for all $p \geq 1$, we have

$$W_p(\mu_t, \mu_V) \to 0 \quad \text{as} \quad t \to \infty.$$

Σ_V: geodesically convex on $\mathscr{P}_2(\mathbb{R})$; lower semi continuous with respect to the Wasserstein topology on $\mathscr{P}(\mathbb{R})$; proper on any geodesic balls of $\mathscr{P}_2(\mathbb{R})$.

Case of uniform convex potential

Using the optimal transportation theory, we can prove that

$$V'' \geq K \Rightarrow \mathrm{Hess}_{W_2}\Sigma_V \geq K.$$

That is to say, the Voiculescu free entropy Σ_V is K-convex on the Wasserstein $\mathscr{P}_2(\mathbb{R})$ if $V'' \geq K$.

Theorem (Li-Li-Xie JSP2020)

Suppose that V is C^2 and $\exists\, K \in \mathbb{R}$ such that $V''(x) \geq K, \quad \forall x \in \mathbb{R}$. Then for all $t > 0$, we have

$$\begin{aligned} \Sigma_V(\mu_t|\mu_V) &\leq e^{-2Kt}\Sigma_V(\mu_0|\mu_V), \\ W_2(\mu_t, \mu_V) &\leq e^{-Kt} W_2(\mu_0, \mu_V). \end{aligned}$$

In particular, if $K > 0$, then μ_t converges to μ_V with the exponential rate K in the W_2-Wasserstein topology on $\mathscr{P}_2(\mathbb{R})$.

Free Log-Sobolev and Talagrand inequalities

To study non-uniform convex but convex case, we need the following free Log-Sobolev inequality and free Talagrand transportation cost inequality.

Theorem (Ledoux-Popescu JFA2009)

Suppose that V is a C^2, convex and there exists a constant $r > 0$ such that

$$V''(x) \geq c > 0, \qquad |x| \geq r.$$

Then there exists a constant $C > 0$ such that the free Log-Sobolev inequality holds: for all probability measure μ with $I_V(\mu) < \infty$ one has

$$\Sigma_V(\mu|\mu_V) \leq \frac{2}{C}\mathrm{I}_V(\mu).$$

Moreover, the free Talagrand transportation inequality holds: there exists a constant $C = C(r, \rho, \mu_V, V) > 0$ such that

$$CW_2^2(\mu, \mu_V) \leq \Sigma_V(\mu|\mu_V).$$

Convex potential which is uniform convex at infinity

Using the above free Log-Sobolev inequality and free Talagrand transportation inequality by Ledoux-Popescu JFA 2009, and applying Otto's argument to the McKean-Vlasov equation, we can prove the following

Theorem (Li-Li-Xie JSP2020)

Suppose that V is a C^2, convex and $\exists\ r > 0$ such that

$$V''(x) \geq K > 0, \quad |x| \geq r.$$

Then there exist two constants $C_1 > 0$ and $C_2 > 0$ such that

$$W_2^2(\mu_t, \mu_V) \leq \frac{e^{-C_1 t}}{C_2} \Sigma_V(\mu_0|\mu), \qquad t > 0.$$

Phase transition

Our theorems can apply to the following case:

- $V(x) = a|x|^p$ with $p > 2$ and $a > 0$.
- the Kontsevich-Penner model

$$V(x) = \frac{ax^4}{12} - \frac{bx^2}{2} - c \log |x|.$$

  Note that, $V''(x) = ax^2 + \frac{c}{x^2} - b \geq 2\sqrt{ac} - b \geq 0$, for all $x \neq 0$ with $a > 0, c > 0$ and $4ac \geq b^2$.

Our theorems can not apply to the double well potential

$$V(x) = ax^4 - bx^2, \quad x \in \mathbb{R},$$

where $a > 0$ and $b > 0$ are constants.
The question whether $\mu_t \to \mu_V$ as $t \to \infty$ for the double well potential or more general non-convex potentials remains open.
If there is phase transition, then

$$\lim_{N\to\infty} \lim_{t\to\infty} L_N(t) \neq \lim_{t\to\infty} \lim_{N\to\infty} L_N(t).$$

Double well potential

Let

$$V(x) = \frac{1}{4}x^4 + \frac{c}{2}x^2, \quad x \in \mathbb{R},$$

where c is a constant. Johansson 1998: the density function of μ_V is given as follows:

- When $c < -2$,

  $$\rho(x) = \frac{1}{2\pi}|x|\sqrt{(x^2 - a^2)(b^2 - x^2)}1_{a<|x|<b},$$

  where $a^2 = -2 - c$ and $b^2 = 2 - c$.
  μ_V has two supports $[-b, -a]$ and $[a, b]$ which are disjoint. By a previous result of Biane-Speicher (2001), μ_t does not converge to μ_V. There is phase transition!

- For $c \in [0, \infty)$, V is C^2 convex and $V''(x) \geq 3$ for $|x| \geq 1$. In this case, $W_2(\mu_t, \mu_V) \to 0$ with exponential convergence rate.

- For $c \in [-2, 0)$, the question whether $W_2(\mu_t, \mu_V) \to 0$ (or even μ_t weakly converges to μ_V) as $t \to \infty$ for the above double-well potential V remains open.

  Conjecture arxiv2013/14 μ_t converges to μ_V weakly and in W_2 as $t \to \infty$.

  There is no phase transition! Donari-Martin, Groux and Maida (arxiv1605.09663, AIHP2018) proved the above conjecture. See also LLX JSP2020 for affirmative result in more general case.

Fluctuation and central limit theorem

By Ito's calculus, for all $f \in C_b^2(\mathbb{R})$, we have

$$\begin{aligned} d\langle L_N(t)-\mu_t, f\rangle &= \frac{1}{N}\sqrt{\frac{2}{\beta N}}\sum_{i=1}^N f'(\lambda_N^i(t))dW_t^i \\ &-\frac{1}{2}\left\langle L_N(t)-\mu_t, V'f'\right\rangle dt+\frac{1}{N}\left(\frac{1}{\beta}-\frac{1}{2}\right)\left\langle L_N(t), f''\right\rangle dt \\ &+\frac{1}{2}\int_{\mathbb{R}^2}\frac{f'(x)-f'(y)}{x-y}(L_N(t,dx)L_N(t,dy)+\mu_t(dx)\mu_t(dy))dt. \end{aligned}$$

Let

$$M_N^f(t) := \sqrt{\frac{1}{N}}\sum_{i=1}^N f'(\lambda_N^i(t))dW_t^i.$$

To study the fluctuation of the McKean-Vlasov limit for $(\mathrm{GDBM})_V$, let us introduce the fluctuation process

$$Y_N(t) = N(L_N(t)-\mu_t).$$

Remark

(1) When $V' = 0$, i.e., in the case of standard Dyson Brownian motion, Anderson, Guionnet and Zeitouni (2010) proved the Functional CLT for $L_N(t)$ with test function $f = x^n$ or more general polynomials.

(2) When $V(x) = \frac{x^2}{2}$ and $\beta = 2$, Israelsson (2001) proved that $Y_N(t)$ is tight and the Stieltjes transform of $Y_N(t)$ converges to the Stieltjes transform of Y_t, which implies that $Y_N(t)$ weakly converges to Y_t. In 2008, Bender extended Israelsson's result for $V(x) = \frac{x^2}{2}$ to all $\beta > 1$.

(3) Dawson 1983 proved FCLT for mean-field model with weak interaction, where the normalized constant is $\sqrt{N}$ as in the usual CLT for i.i.d r.v. Hence GDBM has a very strong interaction and cancellation.

Fluctuation and central limit theorem

Theorem (Li-Xie 2023)

Let

$$Y_N(t) = N(L_N(t) - \mu_t).$$

Suppose that V *satisfies the conditions of* (1), (4) *and* (5), *and* $Y_N(0)$ *converge in distribution in a suitable distribution space* $W_V^{-k,p}(\mathbb{R})$ *to* Y_0. *Then*

$$Y_N(t) \to Y_t \quad \text{in distribution in} \quad C([0,T], W_V^{-k,p}(\mathbb{R})),$$

where Y_t *is a generalized random field valued Gaussian process, and satisfies the following stochastic evolution equation*

$$dY_t = \sqrt{\frac{2}{\beta}} dB_t + \mathcal{A}^*_{\mu_t} Y_t dt + (\frac{1}{\beta} - \frac{1}{2})\mu_t'' dt. \tag{15}$$

Fluctuation and central limit theorem

Here B_t is a $W_V^{-k,p}(\mathbb{R})$-valued Gaussian process of mean zero, whose covariance processes are given by: for any $\varphi_1, \varphi_2 \in C_0^\infty(\mathbb{R})$, and $s, t \in [0, T]$,

$$cov[\langle B_t, \varphi_1 \rangle, \langle B_s, \varphi_2 \rangle] = \int_0^{s \wedge t} \int_{\mathbb{R}} \varphi_1' \varphi_2' \mu_u(dx) du,$$

and for cylinder functional $F : W_V^{-k,p}(\mathbb{R}) \to \mathbb{R}$.

$$\mathcal{A}_{\mu_t} F(\mu) = \left(\frac{1}{2}(V'\mu)' - ((\mathrm{H}\mu_t)\mu)' - ((\mathrm{H}\mu)\mu_t)' \right) \frac{\partial F}{\partial \mu}.$$

H denotes the Hilbert transform on the distributions, $(V'\mu)'$ denotes the derivative of the distribution $V'\mu$ in the sense of distribution, etc.

Let $m_t(f) = \mathbb{E}[\langle Y_t, f \rangle]$. We can derive from (15) that $m_t = \mathbb{E}[\langle Y_t, \cdot \rangle]$ satisfies the following differential equation in the sense of distribution

$$dm_t = \left(\frac{1}{\beta} - \frac{1}{2} \right) \mu_t'' - \frac{1}{2}(m_t V')' + (\mathrm{H}m_t)\mu_t' + (\mathrm{H}\mu_t) m_t',$$

where $\mathrm{H}m_t$ is the Hilbert transform of m_t.

Dyson Brownian motion for Wishart ensemble

Recently, with Rong Lei, we have studied the Dyson Brownian motion associated with the Wishart-Laguerre ensemble on $[0, \infty)^N$

$$\mathbb{P}_{N,M}(x_1, \ldots, x_N) = \frac{1}{Z_{N,M}} \prod_{i<j} |x_i - x_j|^\beta \prod_{j=1}^{N} x_j^{a-p} \exp\left(-N \sum_{i=1}^{N} V(x_i)\right)$$

where $a = \frac{\beta M}{2}$, $p = 1 + \frac{\beta}{2}(N-1)$, $M \geq N$, and $\beta = 1$ for real and $\beta = 2$ for complex.

We can prove the uniqueness of the corresponding McKean-Vlasov equation and the longtime convergence theorem for the McKean-Vlasov equation. Hence we can establish the LLN and Propagation of Chaos for the empirical measure of the Dyson Brownian motion associated with the Wishart-Laguerre ensemble and the Jacobi ensemble. The CLT can be also proved without difficulty. These are included into the PhD thesis of Rong Lei.

Thank you for your attention !